Stochastic Lagrange-Poincaré Reduction

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Outline

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Lagrangians on Lie Groups: Euler-Poincaré Reduction

Let G be a Lie group and $L \in C^\infty(TG)$ be a G-invariant Lagrangian under the tangent lifted left action of G on TG. Given a curve $g(t)$ in G, set $\xi(t) = g(t)^{-1}\dot{g}(t)$. The following are equivalent:

- The variational principle $\delta \int_0^T L(g(t), \dot{g}(t))dt = 0$ holds for variations with fixed endpoints.
- $g(t)$ satisfies the Euler-Lagrange equations

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  are satisfied by $\xi(t)$.

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- We want to reduce the action principle and its associated Euler-Lagrange equations to the dimensionally smaller space TQ/G.
- Given a choice of a principal connection on the bundle $Q \to Q/G$, Cendra, Marsden and Ratiu (2001), showed that

$$\begin{array}{ccc} TQ/G & \xrightarrow{\quad\cong\quad} & T(Q/G) \oplus (Q \times \mathfrak{g})/G \\ & \searrow \qquad \swarrow & \\ & Q/G & \end{array}$$

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- An equivalent formulation of Euler-Lagrange equations is through the Hamilton-Pontryagin (H-P) principle. This is useful for dealing with mechanical systems perturbed by stochastic noise.

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$$\mathcal{A}(q(t), v(t), p(t)) = \int_0^T (L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle) dt$$

over curves $(q(t), v(t), p(t))$ in the **Pontryagin bundle** $TQ \oplus T^*Q$ with $q(0) = a \in Q$ and $q(T) = b \in Q$.

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- This is equivalent to solving the **implicit Euler-Lagrange equations**:

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- The H-P principle is used for developing variational principles for systems with Dirac constraints.

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- Denote by $\tilde{\mathfrak{g}}$ the vector bundle $(Q \times \mathfrak{g})/G \to (Q/G)$, with fibers isomorphic to $\mathfrak{g}$. Here G acts on $\mathfrak{g}$ via the Ad-action.

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- The bundle $(TQ \oplus T^*Q)/G$ decomposes into two parts: a reduced "Euler-Lagrange" part $(T(Q/G) \oplus T^*(Q/G))$ and a "Poincaré" part $(\tilde{\mathfrak{g}} \bigoplus \tilde{\mathfrak{g}}^*)$ (Yoshimura and Marsden, '09).

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- The curvature $B = dA$, which is a $\mathfrak{g}$-valued 2-form, reduces to a $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{B}$ on Q/G. This gives rise to an external force in the reduced Euler-Lagrange equations.

Lagrange-Poincaré Reduction Theorem, Yoshimura and Marsden, '09

The following are equivalent:

- The $TQ \bigoplus T^*Q$-valued curve $(q(t), v(t), p(t))$ is a critical point of the unreduced $H-P$ action functional for variations satisfying $\delta q(t) = 0$ at $t = 0, T$.
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- Let $\ell : T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$ be the reduced Lagrangian. The reduced curve $[q(t), v(t), p(t)]_G \cong (x(t), u(t), y(t), \bar{\eta}(t), \bar{\mu}(t))$ in $T(Q/G) \oplus T^*(Q/G) \oplus \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^*$ is a critical point of the reduced action functional

$$\mathcal{A}^{red} = \int_0^T (\ell(x(t), u(t), \bar{\eta}(t)) + \langle y(t), \dot{x}(t) - u(t) \rangle + \langle \bar{\mu}(t), \bar{\xi}(t) - \bar{\eta}(t))dt$$

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for arbitrary variations $\delta u(t)$, $\delta y(t)$, $\delta\bar{\eta}(t)$ and $\delta\bar{\mu}(t)$ and for constrained variations of the form $\delta x(t) \oplus \delta^A\bar{\xi}(t)$, where
$$\delta^A\bar{\xi}(t) = \frac{D\bar{\zeta}(t)}{Dt} + [\bar{\xi}(t), \bar{\zeta}(t)] + \tilde{B}(\delta x(t), \dot{x}(t))$$
and $\delta x(t)$ vanishes at $t = 0, T$. Here $\bar{\zeta}(t)$ is an arbitrary curve in $\tilde{\mathfrak{g}}$ that vanishes at $t = 0, T$.

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- The reduced curve $(x(t), u(t), \bar{\eta}(t), y(t), \bar{\mu}(t))$ satisfies the following equations:

Horizontal Lagrange-Poincaré Equations

$$\frac{Dy}{Dt} = \frac{\partial \ell}{\partial x} - \langle \bar{\mu}, \tilde{B}(\dot{x}, \cdot)\rangle, \quad y = \frac{\partial \ell}{\partial u}, \quad \dot{x} = u$$

Vertical Lagrange-Poincaré Reduction

$$\frac{D}{Dt}\bar{\mu} = \mathrm{ad}^*_{\bar{\xi}} \bar{\mu}, \quad \bar{\mu} = \frac{\partial \ell}{\partial \bar{\eta}}, \quad \bar{\xi} = \bar{\eta}$$

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Stochastic Hamilton-Pontryagin Principle

- The Hamilton-Pontryagin approach to studying mechanical systems perturbed by random noise was introduced by Bou-Rabee and Owhadi (2008) and studied recenly by Street and Takao (2023).

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- Give a Lagrangian $L \in C^\infty(TQ)$, "noise Lagrangians" $\Gamma_i \in C^\infty(Q)$ $(i = 1, \cdots, k)$ and "noise vector fields" $V_i \in \mathfrak{X}(Q)$, we consider the action functional on $TQ \oplus T^*Q$, given in coordinates (q, v, p) by

$$\mathcal{S}(q_t, v_t, p_t) := \int_0^T L(q_t, v_t)dt + \sum_{i=1}^k \Gamma_i(q_t) \circ dB_t^i + \left\langle p_t, \circ dq_t - v_t dt - \sum_{i=1}^k V_i(q) \circ dB_t^i \right\rangle,$$

where B_t^i is a Brownian motion. We will also assume that Q is endowed with a Riemannian metric and its associated Levi-Civita connection.

Variations of a Semimartingale

Definition

For a semimartingale Γ_t *in a Riemannian manifold* M *with its associated Levi-Civita connection, we consider variations of the form* $\epsilon \mapsto \Gamma_{t,\epsilon}$, *where* $\epsilon \in (-s, s)$ *for some* $s > 0$, *such that:*

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- $\epsilon \mapsto \Gamma_{t,\epsilon}$ *is pathwise smooth and* $\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \Gamma_{t,\epsilon}$ *(the limit in the derivative is taken in the ucp sense) is a semimartingale, denoted by* $\delta\Gamma_t$*.*

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- **Existence**:- Suppose M is geodesically complete. Arnaudon and Thalmaier (1998) show that given a semimartingale Y_t in TM over Γ_t, one can construct a variational family $\Gamma_{t,\epsilon}$ with $\delta\Gamma_t = Y_t$.

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- **Fixed Endpoint Variations**:- Assume that $\Gamma_0 = a$ for some $a \in M$. Let $\|_{0 \to t}^{\Gamma_t} (\cdot)$ denote the stochastic parallel transport along the process Γ_t.

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- **Fixed Endpoint Variations**:- Assume that $\Gamma_0 = a$ for some $a \in M$. Let $\|_{0\to t}^{\Gamma_t}(\cdot)$ denote the stochastic parallel transport along the process Γ_t. If we want the variations to satisfy $\delta\Gamma_0 = \delta\Gamma_T = 0$, we set $Y_t = \|_{0\to t}^{\Gamma_t}(v(t))$, where $v(t)$ is a curve in T_aM with $v(0) = v(T) = 0$. Then, we can construct $\Gamma_{t,\epsilon}$ from Y_t, such that $\delta\Gamma_t = Y_t$. A similar approach has been used in Arnaudon, Chen and Cruzeiro (2014) in the Lie groups context and in Huang and Zambrini (2023) for compact manifolds.

The Stochastic Euler-Lagrange Equations

- Using variations as described, it can be shown that (q_t, v_t, p_t) is a critical point of $\mathcal{S}$ if and only if it satisfies the **stochastic Euler-Lagrange equations**

$$\begin{aligned} \circ dp_t &= \frac{\partial L}{\partial q_t} dt + \sum_{i=1}^{k} \left(\frac{\partial \Gamma_i}{\partial q_t} - \frac{\partial}{\partial q_t} \langle p_t, V_i \rangle \right) \circ dB_t^i \\ p_t &= \frac{\partial L}{\partial v_t} \\ \circ dq_t &= v_t dt + \sum_{i=1}^{k} V_i(q_t) \circ dB_t^i. \end{aligned}$$

Outline

A Symmetry Condition on Noise Vector Fields

- We assume that the noise vector fields V_i satisfy the following condition: Let $Pr : TQ \to TQ/G$ denote the projection. There exists vector fields Θ_i on Q/G and constants $\beta_i \in \mathfrak{g}$ such that if θ_i denotes the section $[q] \mapsto [q, \beta_i]_G = \bar{\beta}_i$ of $\tilde{\mathfrak{g}}$ then $Pr \circ V_i \cong (\Theta_i \oplus \theta_i) \circ \pi$.

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- Let $\ell : T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$ and $\gamma_i : Q/G \to \mathbb{R}$ denote the reduced Lagrangian and noise Lagrangians respectively.

Stochastic Lagrange-Poincaré Reduction Theorem, S., '24

The following statements are equivalent:

- The $TQ \oplus T^*Q$-valued semimartingale (q_t, v_t, p_t) is a critical point for the action functional $\mathcal{S}$ for variations such that δv_t and δp_t are arbitrary and $\delta q_t = 0$ at $t = 0, T$.
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- The semimartingale $[q_t, v_t, p_t]_G = (x_t, u_t, y_t, \bar{\eta}_t, \bar{\mu}_t)$ extremizes the reduced action functional $\mathcal{S}^{red}$

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- The semimartingale $(x_t, u_t, y_t, \bar{\eta}_t, \bar{\mu}_t)$ satisfies the following equations:

Horizontal Stochastic Lagrange-Poincaré Equations

$$\circ Dy_t = \frac{\partial \ell}{\partial x_t} dt + \sum_{i=1}^{k} \left(\frac{\partial \gamma_i}{\partial x_t} - \frac{\partial}{\partial x_t} \langle y_t, \Theta_i(x_t) \rangle \right) \circ dB_t^i$$
$$- \langle \bar{\mu}_t, \tilde{B}(\circ dx_t, \cdot) \rangle,$$
$$y_t = \frac{\partial \ell}{\partial u_t},$$
$$\circ dx_t = u_t dt + \sum_{i=1}^{k} \Theta_i(x_t) \circ dB_t^i.$$

Vertical Stochastic Lagrange-Poincaré Reduction

$$\circ D\bar{\mu}_t = \mathrm{ad}^*_{\circ d\bar{\xi}_t} \bar{\mu}_t$$
$$\circ d\bar{\xi}_t = \bar{\eta}_t dt + \sum_{i=1}^{k} \bar{\beta}_i \circ dB_t^i$$
$$\bar{\mu}_t = \frac{\partial \ell}{\partial \bar{\eta}_t}.$$

Special Cases

- $Q = G$: In this case, the horizontal stochastic Lagrange-Poincaré equations vanish and the vertical stochastic Lagrange-Poincaré equations are the stochastic Euler-Poincaré equations.
- $G = \{e\}$: In this case the vertical stochastic Lagrange-Poincaré equations vanish and the horizontal stochastic Lagrange-Poincaré equations become the stochastic Euler-Lagrange equations.

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- The Horizontal Noise Case: Set $\bar{\beta}_i = 0$. Then, the vertical Lagrange-Poincaré equations are noise-free and agree with deterministic vertical Lagrange-Poincaré equations.
- The Vertical Noise Case: Set $\Gamma_i = 0$ and $\Theta_i = 0$. Then, the horizontal Lagrange-Poincaré equations become

$$
\begin{aligned}
\circ D y_t &= \frac{\partial \ell}{\partial x_t} dt - \langle \bar{\mu}_t, \tilde{B}(\dot{x}_t, \cdot) \rangle, \\
y_t &= \frac{\partial \ell}{\partial u_t}, \\
\dot{x}_t &= u_t
\end{aligned}
$$

which agree with the deterministic horizontal Lagrange-Poincaré equations up to a stochastic forcing term given by $\langle \bar{\mu}_t, \tilde{B}(\dot{x}_t, \cdot) \rangle$.

Outline

The Deterministic Kaluza-Klein Approach to Charged Particles

- The equation for a charged particle in a magnetic field $\mathbf{B}$ is given by $\dot{\mathbf{v}} = \frac{e}{c}\mathbf{v} \times \mathbf{B}$. It can be viewed as a reduction of the geodesic flow on $Q_K = \mathbb{R}^3 \times S^1$ under a certain metric (Marsden and Ratiu, '98).

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- Let $G = S^1$ with its standard bi-invariant metric κ and consider $\mathbb{R}^3$ with its standard metric given by the inner product $\langle , \rangle$.
- Let $\mathbf{A}$ be a vector in $\mathbb{R}^3$ and identify $\mathbf{A}$ with a 1-form A on $\mathbb{R}^3$. Let

$$\alpha = A + d\theta$$

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- Consider the metric on Q_K given by

$$g((\mathbf{u}_q, u_\theta), (\mathbf{v}_q, v_\theta)) = \langle \mathbf{u}_q, \mathbf{v}_q \rangle + \kappa(\alpha(\mathbf{u}_q, v_\theta), \alpha(\mathbf{v}_q, v_\theta)).$$

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- The Lagrangian for the geodesic flow on (Q_K, g) is given by

$$L(q, \theta, \mathbf{v}_q, v_\theta) = \frac{1}{2}\left(|\mathbf{v}_q|^2 + (\mathbf{A} \cdot \mathbf{v}_q + v_\theta)^2\right).$$

We will call it the **Kaluza-Klein Lagrangian**.

Reduction of the Kaluza-Klein Lagrangian

- Let $B = d\alpha = dA$ and identify B with the vector $\mathbf{B} = \nabla \times \mathbf{A}$. The reduced curvature 2-form on $Q_K/S^1 \cong \mathbb{R}^3$ is identified with B or $\mathbf{B}$.

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- Let $(\mathbf{x}, \mathbf{u}, \lambda) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ denote local coordinates on the bundle $T\mathbb{R}^3 \oplus \tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}}$ is the associated bundle. The reduced Lagrangian is

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$$\dot{p}_\theta = 0, p_\theta = \lambda.$$

  Here p_θ is the momentum conjugate to λ and is given by

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- The horizontal Lagrange-Poincaré equations become

$$\dot{\mathbf{u}}_q = \frac{e}{c}(\mathbf{u} \times \mathbf{B}),$$

  which is the Lorentz force law.

The Stochastic Version

- Let L denote the Kaluza-Klein Lagrangian. Let $\Gamma \in C^\infty(Q_K)$ and

$$V(\mathbf{q}, \theta) = (\mathbf{V}(\mathbf{q}), \Psi(\theta)) \in T_{(\mathbf{q},\theta)} Q_K$$

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Thank You